Stability of the superposition of boundary layer and rarefaction wave for outflow problem on the two-fluid Navier-Stokes-Poisson system

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Abstract

This paper is concerned with the study of nonlinear stability of superposition of boundary layer and rarefaction wave on the two-fluid Navier-Stokes-Poisson system in the half line $\mathbb{R}_+ =: (0, +\infty)$. On account of the quasineutral assumption and the absence of the electric field for the large time behavior, we successfully construct the boundary layer and rarefaction wave, and then we give the rigorous proofs of the stability theorems on the superposition of boundary layer and rarefaction wave under small perturbations for the corresponding initial boundary value problem of the Navier-Stokes-Poisson system, only provided the strength of boundary layer is small while the strength of rarefaction wave can be arbitrarily large. The complexity of nonlinear composite wave leads to many complicated terms in the course of establishing the *a priori* estimates. The proofs are given by an elementary L^2 energy method.

Key words. Navier-Stokes-Poisson; boundary layer and rarefaction wave; stability.

AMS subject classifications. 35Q35, 35B40, 35B45.

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1 Introduction

1.1 The problem

The dynamics of the charged particles in the collisional dusty plasma can be described by the Navier-Stokes-Poisson (called NSP in the sequel for simplicity) system which reads in the Eulerian coordinates

$$
\begin{cases}
\partial_t \rho_i + \partial_x(\rho_i u_i) = 0, \\
\rho_i(\partial_t u_i + u_i \partial_x u_i) + \partial_x P(\rho_i) = \rho_i E + \mu_i \partial_x^2 u_i, \\
\partial_t \rho_e + \partial_x(\rho_e u_e) = 0, \\
\rho_e(\partial_t u_e + u_e \partial_x u_e) + \partial_x P(\rho_e) = -\rho_e E + \mu_e \partial_x^2 u_e, \\
\partial_x E = \rho_i - \rho_e.
\end{cases}
\tag{1.1}
$$

Here, for $\alpha = i, e$, $P(\rho_\alpha)$ is pressure which is given by

$$
P(\rho_\alpha) = A\rho_\alpha^{\gamma_\alpha}, \tag{1.2}
$$

where A is a positive constant and $\gamma_\alpha > 1$ is the adiabatic exponent. Thus each fluid (ions or electrons) is regarded as an ideal polytropic gas. The unknown functions ρ_α and u_α stand for the density and velocity of ions ($\alpha = i$) and electrons ($\alpha = e$) in plasma, respectively, and E is the electric field, while the positive constants $\mu_\alpha > 0$ denote the viscosity coefficient of ions ($\alpha = i$) and electrons ($\alpha = e$), respectively. Throughout the paper, for brevity we assume $\gamma_i = \gamma_e = \gamma > 1$; the case of $\gamma_i \neq \gamma_e$ and $\gamma_i = \gamma_e = 1$ could be considered in a similar way. We also assume $\mu_i = \mu_e = 1$ throughout the paper. One can see [1] and [13] for more information about the physical background of model (1.1).

We consider (1.1) in the half line $\mathbb{R}_+$ with initial data

$$
[\rho_i, u_i, \rho_e, u_e](x, 0) = [\rho_{i0}, u_{i0}, \rho_{e0}, u_{e0}](x) \to [\rho_+, u_+, \rho_+, u_+] \quad \text{as} \quad x \to +\infty, \tag{1.3}
$$

where $\rho_+ > 0$ and u_+ are constants. The boundary conditions are

$$
u_i(0, t) = u_e(0, t) = u_b < 0, \quad \forall\, t \geq 0, \tag{1.4}
$$

and the compatibility condition $u_b = u_{i0}(0) = u_{e0}(0)$ holds.

In the case of $u_b < 0$, electrons and ions fluids flow away from the boundary $\{x = 0\}$, and thus the problem (1.1), (1.3) and (1.4) in such case is called an outflow problem. The case of $u_b = 0$ and $u_b > 0$ is called the impermeable wall problem and the inflow problem, respectively. Notice that for the inflow problem, there should been an additional boundary condition on the density. In the paper, we focus on the outflow problem in the case of $u_b < 0$. Here we remark that the impermeable wall problem and the inflow problem of the Navier-Stokes-Poisson system are left for study in the future.

1.2 Some preliminary

In order to study the large time behavior of solutions to the initial boundary value problem (1.1), (1.3) and (1.4), we notice that in the simplified case of the electric field $E = 0$ and the quasineutral assumptions $\rho_i = \rho_e$ and $u_i = u_e$ for the large time behavior, the problem is reduced to consider the following single quasineutral Navier-Stokes equation

$$
\begin{cases}
\partial_t \rho + \partial_x(\rho u) = 0, \\
\rho(\partial_t u + u\partial_x u) + \partial_x P(\rho) = \partial_x^2 u
\end{cases}
\tag{1.5}
$$

with initial data

$$
[\rho, u](x, 0) = [\rho_0, u_0](x) \to [\rho_+, u_+], \quad as \quad x \to +\infty \tag{1.6}
$$

and the boundary condition

$$u(0,t) = u_b < 0, \quad \forall t \geq 0. \tag{1.7}$$

Matsumura [14] gave the classification of the large time behavior solutions to the outflow problem for Navier-Stokes equation (1.5) in terms of (ρ_+, u_+) and $u_b < 0$. In what follows, let us recall some basic facts concerning the study of the outflow problem. The characteristic speeds of the hyperbolic part of (1.5) are

$$\lambda_1 = u - C(\rho), \quad \lambda_2 = u + C(\rho), \tag{1.8}$$

where $C(\rho) = \sqrt{P'(\rho)} = \sqrt{\gamma A}\rho^{\frac{\gamma-1}{2}}$ is the local sound speed. From now on, we define

$$v = \frac{1}{\rho}, \quad v_+ = \frac{1}{\rho_+}, \cdots, \textit{and so on},$$

where v is the specific volume. Let

$$C_+ = C(\rho_+) = \sqrt{\gamma A}\rho_+^{\frac{\gamma-1}{2}} = \sqrt{\gamma A}v_+^{-\frac{\gamma-1}{2}}, \quad M_+ = \frac{|u_+|}{C_+}$$

be the sound speed and the Mach number at the far field $x = +\infty$, respectively. The phase plane $\mathbb{R}_+ \times \mathbb{R}$ of (v,u) can be divided into three subsets:

$$\Omega_{sub} := \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; \quad |u| < C\left(\frac{1}{v}\right)\right\},$$

$$\Gamma_{trans} := \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; \quad |u| = C\left(\frac{1}{v}\right)\right\},$$

$$\Omega_{super} := \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; \quad |u| > C\left(\frac{1}{v}\right)\right\},$$

where Ω_{sub}, Γ_{trans} and Ω_{super} are called the subsonic, transonic and supersonic regions, respectively. In the phase plane, we denote the curves through a right state point (v_1, u_1):

$$BL(v_1,u_1) = \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; \quad \frac{u}{v} = \frac{u_1}{v_1}\right\},$$

$$R_2(v_1,u_1) = \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; \quad u = u_1 - \sqrt{\gamma A}\int_{v_1}^{v} s^{-\frac{\gamma+1}{2}}ds, \quad v > v_1\right\},$$

$$S_2(v_1,u_1) = \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R}; \quad u = u_1 + \sqrt{\left[P\left(\frac{1}{v}\right) - P\left(\frac{1}{v_1}\right)\right](v_1 - v)}, \quad v < v_1\right\},$$

to be the boundary line, 2-rarefaction wave and 2-shock wave curves, respectively. Then the large time behavior of solutions to the outflow problem (1.5), (1.6) and (1.7) can be classified into the following four cases (the cases are omitted which concern shock waves):

Case I: $(v_+, u_+) \in \Omega_{super}\bigcap\{u_+ < 0\}$ and $u_b < u_*$. Here (v_*, u_*) is an intersection point of $BL(v_+, u_+)$ and $S_2(v_+, u_+)$, ie.,

$$u_+ = \frac{u_+}{v_+}v_* - \sqrt{\left[P\left(\frac{1}{v_*}\right) - P\left(\frac{1}{v_+}\right)\right](v_+ - v_*)}, \quad u_* = \frac{u_+}{v_+}v_*. \tag{1.9}$$

Then there exists a unique v_b such that $(v_b, u_b) \in BL(v_+, u_+)$, and the time asymptotic state of solution is a boundary layer $(\tilde{v}, \tilde{u})(x)$ which connects (v_b, u_b) with (v_+, u_+), see Figure 1. By the relation of ρ and v, then we can say that boundary layer $(\tilde{\rho}, \tilde{u})(x)$ connects (ρ_b, u_b) with (ρ_+, u_+). The boundary layer $(\tilde{\rho}, \tilde{u})(x)$ will be explained in next section.

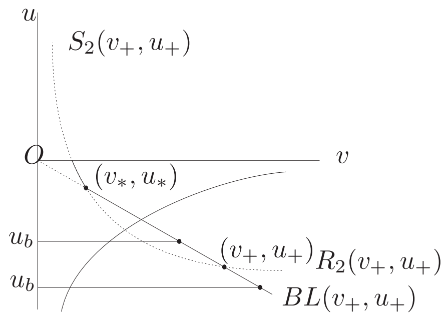

Figure 1

Case II: $(v_+, u_+) \in \Gamma_{trans} \bigcap \{u_+ < 0\}$ and $u_b < u_+$. Then there exists a unique v_b such that $(v_b, u_b) \in BL(v_+, u_+)$, and the time-asymptotic state of solution is a boundary layer $(\tilde{v}, \tilde{u})(x)$ which connects (v_b, u_b) with (v_+, u_+), see Figure 2. Here, the boundary layer $(\tilde{v}, \tilde{u})(x)$ is degenerate. That is to say boundary layer $(\tilde{\rho}, \tilde{u})(x)$ connects (ρ_b, u_b) with (ρ_+, u_+), and the boundary layer $(\tilde{\rho}, \tilde{u})(x)$ is degenerate.

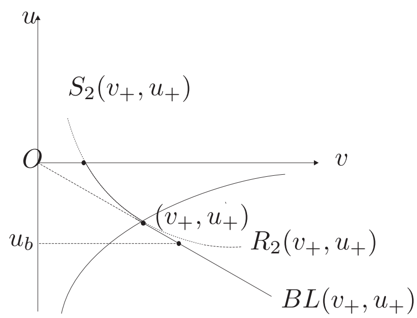

Figure 2

Case III: $(v_+, u_+) \in \Omega_{sub} \bigcap \{u_+ < 0\}$ and $u_b < u_+$. Here (v_*, u_*) is an intersection point of $R_2(v_+, u_+)$ and Γ_{trans}, ie.,

$$u_+ - \sqrt{\gamma A} \int_{v_+}^{v_*} s^{-\frac{\gamma+1}{2}} ds = -\sqrt{\gamma A} v_*^{-\frac{\gamma-1}{2}}, \quad u_* = -\sqrt{\gamma A} v_*^{-\frac{\gamma-1}{2}}, \tag{1.10}$$

see Figure 3. This case is divided into two subcases:

Subcase 1: If $u_* \leq u_b < u_+$, then there exists a unique v_b such that $(v_b, u_b) \in R_2(v_+, u_+)$, and the time-asymptotic state of solution is a 2-rarefaction wave $(v^{R_2}, u^{R_2})(\frac{x}{t})$, which connects (v_b, u_b) with (v_+, u_+), to the corresponding Riemann problem, while the 2-rarefaction wave $(\rho^{R_2}, u^{R_2})(\frac{x}{t})$ connects (ρ_b, u_b) with (ρ_+, u_+).

Subcase 2: If $u_* > u_b$, then there exists a unique v_b such that $(v_b, u_b) \in BL(v_*, u_*)$, and the time-asymptotic state of solution is the superposition of a boundary layer $(\tilde{v}, \tilde{u})(x)$ connecting (v_b, u_b) with (v_*, u_*), which is degenerate, and a 2-rarefaction wave $(v^{R_2}, u^{R_2})(\frac{x}{t})$ connecting (v_*, u_*) with (v_+, u_+), while boundary layer $(\tilde{\rho}, \tilde{u})(x)$ connects (ρ_b, u_b) with (ρ_*, u_*), and a 2-rarefaction wave $(\rho^{R_2}, u^{R_2})(\frac{x}{t})$ connects (ρ_*, u_*) with (ρ_+, u_+).

Case IV: $u_+ > 0$ and $u_b < 0$. Here (v_*, u_*) is an intersection point of $R_2(v_+, u_+)$ and Γ_{trans} which is defined by (1.10), see Figure 4. This case is divided into two subcases:

Subcase 1: If $u_* \leq u_b < 0$, then there exists a unique v_b such that $(v_b, u_b) \in R_2(v_+, u_+)$, and the time-asymptotic state of solution is a 2-rarefaction wave $(v^{R_2}, u^{R_2})(\frac{x}{t})$, which connects (v_b, u_b) with (v_+, u_+),

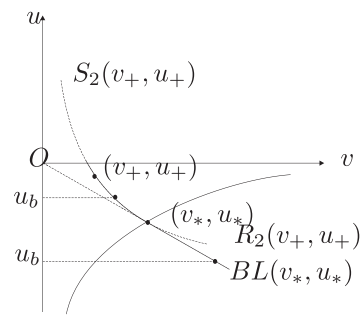

Figure 3

to the corresponding Riemann problem, while a 2-rarefaction wave $(\rho^{R_2}, u^{R_2})(\frac{x}{t})$ connects (ρ_b, u_b) with (ρ_+, u_+).

Subcase 2: If $u_* > u_b$, then there exists a unique v_b such that $(v_b, u_b) \in BL(v_*, u_*)$, and the time-asymptotic state of solution is the superposition of a boundary layer $(\tilde{v}, \tilde{u})(x)$ connecting (v_b, u_b) with (v_*, u_*), which is degenerate, and a 2-rarefaction wave $(v^{R_2}, u^{R_2})(\frac{x}{t})$ connecting (v_*, u_*) with (v_+, u_+), while boundary layer $(\tilde{\rho}, \tilde{u})(x)$ connects (ρ_b, u_b) with (ρ_*, u_*), and a 2-rarefaction wave $(\rho^{R_2}, u^{R_2})(\frac{x}{t})$ connects (ρ_*, u_*) with (ρ_+, u_+).

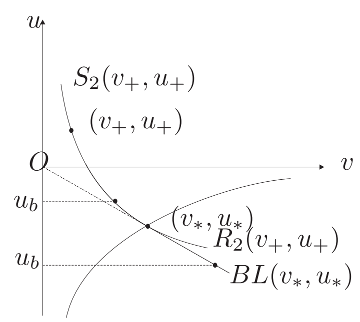

Figure 4

1.3 Boundary layer and rarefaction wave

In the paper, we study the subcase 2 in Case III or Case IV without considering the other cases since the cases of the single wave have been studied by Duan and Yang [6]. Recalling subcase 2 in Case III or Case IV, there exists a unique v_b in phase plane such that $(v_b, u_b) \in BL(v_*, u_*)$, where (v_*, u_*) is defined in (1.10). And the solution to the initial boundary value problem (1.1), (1.3) and (1.4) for the outflow problem on two-fluid Navier-Stokes-Poisson system is expected to tend to the superposition of a degenerate boundary layer $(\tilde{\rho}, \tilde{u})(x)$ connecting (ρ_b, u_b) with (ρ_*, u_*) and a 2-rarefaction wave $(\rho^{R_2}, u^{R_2})(\frac{x}{t})$ connecting (ρ_*, u_*) with (ρ_+, u_+) as $t \to +\infty$ coupling the trivial profile of electric field $E = 0$.

First of all, we define the boundary layer $(\tilde{\rho}, \tilde{u})$ by the stationary solution to

$$\begin{cases} \partial_x(\tilde{\rho}\tilde{u}) = 0, \quad x \in \mathbb{R}_+, \\ \tilde{\rho}\tilde{u}\partial_x\tilde{u} + \partial_x P(\tilde{\rho}) = \partial_x^2 \tilde{u}, \quad x \in \mathbb{R}_+, \\ \tilde{u}(0) = u_b, \quad (\tilde{\rho}, \tilde{u})(+\infty) = (\rho_*, u_*), \quad \inf\limits_{x \in \mathbb{R}_+} \tilde{\rho}(x) > 0. \end{cases} \tag{1.11}$$

Integrating $(1.11)_1$ over $[x, +\infty)$ for $x > 0$, and letting $x \to 0$, we obtain the value of $\tilde{\rho}(x)$ at the boundary

$\{x=0\}$ as follows:

$$\rho_b := \tilde{\rho}(0) = \frac{\rho_* u_*}{u_b}. \tag{1.12}$$

Since $u_b < 0$, we have $u_* < 0$. The strength of the boundary layer $(\tilde{\rho}, \tilde{u})(x)$ is measured by

$$\tilde{\delta} := |u_* - u_b|. \tag{1.13}$$

In what follows let us present the existence and some known properties of the boundary layer $(\tilde{\rho}, \tilde{u})(x)$ connecting (ρ_b, u_b) with (ρ_*, u_*) for the stationary problem (1.11). Here we only list the properties of the degenerate boundary layer. Please refer to [9] or [14] for details.

Lemma 1.1. *By the definition of (v_*, u_*) in Subcase 2 in Case III or Case IV (i.e. it is located at the transonic curve), then there exists a solution $(\tilde{\rho}, \tilde{u})(x)$ to the stationary problem (1.11) such that $\tilde{u} = \frac{u_*}{v_*}\tilde{v}$, $\tilde{v} = \frac{1}{\tilde{\rho}}$. Moreover, $\tilde{u}(x)$ is monotonically increasing ($\partial_x \tilde{u} \geq 0$) and converges to u_* algebraically as x tends to infinity. Precisely, there exists a positive constant C such that*

$$|\partial_x^k[\tilde{\rho} - \rho_*, \tilde{u} - u_*]| \leq \frac{C\tilde{\delta}^{k+1}}{(1+\tilde{\delta}x)^{k+1}}, \quad k = 0, 1, 2, \cdots. \tag{1.14}$$

Since the 2-rarefaction wave $\left[\rho^{R_2}, u^{R_2}\right](\frac{x}{t})$ is a weak solution, we shall construct a smooth approximation for the 2-rarefaction wave above in the following. Firstly, consider the Riemann problem for Burger's equation:

$$\begin{cases} \partial_t w + w\partial_x w = 0, \\ w(0,x) = w_0(x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0, \end{cases} \end{cases} \tag{1.15}$$

where $w_- < w_+$. Then it is well known that (1.15) has a continuous weak solution $w^{R_2}(\frac{x}{t})$ whose explicit form is given by

$$w^{R_2}(\frac{x}{t}) = \begin{cases} w_-, & x < w_- t, \\ \dfrac{x}{t}, & w_- t \leq x \leq w_+ t, \\ w_+, & x > w_+ t. \end{cases} \tag{1.16}$$

Moreover, $w^{R_2}(\frac{x}{t})$ can be approximated by the smooth function $\overline{w}(t,x)$ which is a solution to

$$\begin{cases} \partial_t \bar{w} + \bar{w}\partial_x \bar{w} = 0, \\ \bar{w}(0,x) = \bar{w}_0(x) = \begin{cases} w_-, & x < 0, \\ w_- + C_q\bar{\delta}\displaystyle\int_0^{\epsilon x} y^q e^{-y} dy, & x > 0, \end{cases} \end{cases} \tag{1.17}$$

where $\bar{\delta} := w_+ - w_-$, $q \geq 10$ is a constant, C_q is a constant such that $C_q \int_0^\infty y^q e^{-y} dy = 1$, and $\epsilon \leq 1$ is a positive constant to be determined later. Then we have the following lemma.

Lemma 1.2. *Let $\bar{\delta} = w_+ - w_-$ be the wave strength of the 2-rarefaction wave. Then the problem (1.17) has a unique smooth solution $\bar{w}(x,t)$ which satisfies the following properties:*
(i) $0 < w_- < \bar{w}(x,t) < w_+$, $\partial_x \bar{w} \geq 0$ for $x \in \mathbb{R}$ and $t \geq 0$.
(ii) For any p ($1 \leq p \leq +\infty$), there exists a constant $C_{p,q}$ such that for $t \geq 0$

$$\|\partial_x \bar{w}\|_{L^p} \leq C_{p,q} \min\{\bar{\delta}\epsilon^{1-\frac{1}{p}}, \bar{\delta}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\},$$

$$\|\partial_x^2 \bar{w}\|_{L^p} \leq C_{p,q} \min\{\bar{\delta}\epsilon^{2-\frac{1}{p}}, \bar{\delta}^{\frac{1}{q}} \epsilon^{1-\frac{1}{p}+\frac{1}{q}} t^{-1+\frac{1}{q}}\}.$$

(iii) When $x \leq w_- t$, $\bar{w} - w_- = \partial_x \bar{w} = \partial_x^2 \bar{w} = 0$.
(iv) $\displaystyle\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |\bar{w}(x,t) - w^{R_2}(\tfrac{x}{t})| = 0$.

Then the smooth approximate rarefaction wave $[\rho^{r_2}, u^{r_2}](x,t)$ which corresponds to the rarefaction wave $\left[\rho^{R_2}, u^{R_2}\right](\frac{x}{t})$ can be defined as follows:

$$\begin{cases} u^{r_2} + C(\rho^{r_2}) = \bar{w}(x, 1+t), \ w_- = u_* + C(\rho_*) = 0, \quad w_+ = u_+ + C(\rho_+) > 0, \\ u^{r_2} = u_+ - \sqrt{\gamma A} \int_{v_+}^{v^{r_2}} s^{-\frac{\gamma+1}{2}} ds, \quad v^{r_2} = \frac{1}{\rho^{r_2}}, \quad v_+ = \frac{1}{\rho_+}, \end{cases} \tag{1.18}$$

where $\bar{w}(x,t)$ is given in (1.17).

It is easy to obtain $[\rho^{r_2}, u^{r_2}](x,t)$ satisfies

$$\begin{cases} \partial_t \rho^{r_2} + \partial_x(\rho^{r_2} u^{r_2}) = 0, \\ \rho^{r_2} \partial_t u^{r_2} + \rho^{r_2} u^{r_2} \partial_x u^{r_2} + \partial_x P(\rho^{r_2}) = 0. \end{cases} \tag{1.19}$$

Here we restrict $[\rho^{r_2}, u^{r_2}](x,t)$ in the half space $\{x \geq 0\}$. Then one has

Lemma 1.3. *Let $\delta_r = |\rho_+ - \rho_*| + |u_+ - u_*|$ be the wave strength of the 2-rarefaction wave. Then the smooth approximate 2-rarefaction wave $[\rho^{r_2}, u^{r_2}](x,t)$ constructed in (1.18) has the following properties:*
(i) $\partial_x u^{r_2} \geq 0$, $\rho_* < \rho^{r_2}(x,t) < \rho_+$, $u_* < u^{r_2}(x,t) < u_+$, $\partial_x u^{r_2} \sim |\partial_x \rho^{r_2}|$ for $x \in \mathbb{R}_+$ and $t \geq 0$.
(ii) For any p $(1 \leq p \leq +\infty)$, there exists a constant $C_{p,q}$ such that for $t > 0$,

$$\|\partial_x[\rho^{r_2}, u^{r_2}]\|_{L^p(\mathbb{R}_+)} \leq C_{p,q} \min\{\delta_r \epsilon^{1-\frac{1}{p}}, \delta_r^{\frac{1}{p}} (1+t)^{-1+\frac{1}{p}}\},$$

$$\|\partial_x^2[\rho^{r_2}, u^{r_2}]\|_{L^p(\mathbb{R}_+)} \leq C_{p,q} \min\{\delta_r \epsilon^{2-\frac{1}{p}}, \delta_r^{\frac{1}{q}} \epsilon^{1-\frac{1}{p}+\frac{1}{q}} (1+t)^{-1+\frac{1}{q}}\}.$$

(iii) $[\rho^{r_2}, u^{r_2}](0,t) = [\rho_*, u_*]$.
(iv) $\lim\limits_{t\to+\infty} \sup\limits_{x\in\mathbb{R}_+} \left|[\rho^{r_2}, u^{r_2}](x,t) - \left[\rho^{R_2}, u^{R_2}\right](\frac{x}{t})\right| = 0$.

Now, we define

$$[\hat{\rho}, \hat{u}](x,t) := [\tilde{\rho}, \tilde{u}](x) + [\rho^{r_2}, u^{r_2}](x,t) - [\rho_*, u_*]. \tag{1.20}$$

By a straightforward calculation, we have

$$\begin{cases} \partial_t \hat{\rho} + \partial_x(\hat{\rho}\hat{u}) = \hat{f}, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ \hat{\rho}(\partial_t \hat{u} + \hat{u}\partial_x \hat{u}) + \partial_x P(\hat{\rho}) = \partial_x^2 \hat{u} + \hat{g}, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (\hat{\rho}, \hat{u})(x,0) \to (\rho_+, u_+), \quad \text{as} \quad x \to +\infty, \quad (\hat{\rho}, \hat{u})(0,t) = (\rho_b, u_b). \end{cases} \tag{1.21}$$

where

$$\begin{cases} \hat{f} = \partial_x \tilde{\rho}(u^{r_2} - u_*) + \partial_x \tilde{u}(\rho^{r_2} - \rho_*) + \partial_x \rho^{r_2}(\tilde{u} - u_*) + \partial_x u^{r_2}(\tilde{\rho} - \rho_*), \\ \hat{g} = -\partial_x^2 u^{r_2} + \tilde{u}\partial_x \tilde{u}(\rho^{r_2} - \rho_*) + \hat{\rho}\left[\partial_x \tilde{u}(u^{r_2} - u_*) + \partial_x u^{r_2}(\tilde{u} - u_*)\right] \\ \qquad + \partial_x \tilde{\rho}\left[P'(\hat{\rho}) - P'(\tilde{\rho})\right] + \partial_x \rho^{r_2}\left[P'(\hat{\rho}) - P'(\rho^{r_2})\right] - \dfrac{P'(\rho^{r_2})}{\rho^{r_2}} \partial_x \rho^{r_2}(\tilde{\rho} - \rho_*). \end{cases} \tag{1.22}$$

From $(1.11)_1$ and (1.20), it is easy to know

$$\begin{cases} |\hat{f}| + |\hat{g} + \partial_x^2 u^{r_2}| \leq C\left\{\partial_x \tilde{u}(u^{r_2} - u_*) + \partial_x u^{r_2}(u_* - \tilde{u})\right\}, \\ |\partial_x \hat{f}| \leq C\left\{(|\partial_x^2 \tilde{u}| + (\partial_x \tilde{u})^2)(u^{r_2} - u_*) + \partial_x \tilde{u}\partial_x u^{r_2} + |\partial_x^2 u^{r_2}| + (\partial_x u^{r_2})^2\right\}, \end{cases} \tag{1.23}$$

where $\partial_x \tilde{u} \geq 0$, $\partial_x u^{r_2} \geq 0$ and $\tilde{u} \leq u_* \leq u^{r_2}$.

1.4 Main results

We can easily derive $E(x,t) = -\int_x^{+\infty}[\rho_i(y,t) - \rho_e(y,t)]dy$ from $(1.1)_5$ if we assume that $E(x,t) \to 0$ as $x \to +\infty$ holds. Then we can define $E(x,0) = -\int_x^{+\infty}[\rho_{i0}(y) - \rho_{e0}(y)]dy$. Now we are in a position to state our main results.

Theorem 1.1. *Let $\alpha = i, e$ and assume that constant states u_b, u_* and the infinite state (ρ_+, u_+) satisfy Subcase 2 either in Case III or in Case IV. There exist some positive constants $\varepsilon_0 > 0$ and $C_0 > 0$ such that if*

$$\|[\rho_{\alpha 0}(\cdot) - \hat{\rho}(\cdot,0), u_{\alpha 0}(\cdot) - \hat{u}(\cdot,0)]\|_{H^1}^2 + \|E(\cdot,0)\|^2 + \epsilon^{\frac{1}{10}} + \tilde{\delta}^{\frac{1}{9}} \le \varepsilon_0^2, \tag{1.24}$$

where $\epsilon > 0$ is the parameter appearing in (1.17)*, then the initial boundary value problem* (1.1)*,* (1.3) *and* (1.4) *admits a unique global solution $[\rho_\alpha, u_\alpha, E](x,t)$ satisfying*

$$\sup_{t \ge 0} \|[\rho_\alpha - \hat{\rho}, u_\alpha - \hat{u}, E](\cdot,t)\|_{H^1} \le C_0 \varepsilon_0. \tag{1.25}$$

Moreover, the solution $[\rho_\alpha, u_\alpha, E](x,t)$ tends time-asymptotically to the composite wave in the sense that

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}_+} \left| [\rho_\alpha, u_\alpha](x,t) - [\tilde{\rho}, \tilde{u}](x) - \left[\rho^{R_2}, u^{R_2}\right]\left(\frac{x}{t}\right) + [\rho_*, u_*] \right| = 0, \tag{1.26}$$

and

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}_+} |E| = 0. \tag{1.27}$$

As it is well known that, there have been a great number of mathematical studies about the outflow problem, impermeable wall problem and inflow problem of Navier-Stokes system, please referring to [7, 8, 9, 10, 15, 16] and the references therein. However, to the best of our knowledge, there are very few results about the above mentioned problems for NSP system. Duan-Yang [6] firstly proved the stability of rarefaction wave and boundary layer for outflow problem on the two-fluid NSP system. One important point used in [6] is that the large time behavior of the electric fields is trivial and hence the two fluids indeed have the same asymptotic profiles which are constructed from the Navier-Stokes equations without any force under the assumptions that all physical parameters in the model must be unit, which is obviously impractical since ions and electrons generally have different masses. The convergence rate of corresponding solutions toward the stationary solution was obtained by Zhou-Li [20]. In the paper, we study the nonlinear stability of the superposition of boundary layer and rarefaction wave for outflow problem on two-fluid NSP system. The complexity of nonlinear composite wave leads to many complicated terms in the course of establishing the *a priori* estimates. Lemma 4.1 plays crucial role to deal with the complicated terms. Compared with Navier-Stokes system, the key to prove Theorem 1.1 for NSP system is to deal with the extra electric field E which is no longer L^2 integrate in space and time due to the structure of the Poisson equation in $(2.2)_5$. The detailed way to deal with the terms involved with electric field E is stated in (2.12),(2.13) and (2.15). Finally, we remark that NSP system (1.1) in the non-dimensional form depends generally on the ratios of masses, charges and temperatures of two fluids. If we don't ignore these physical coefficients, the two-fluid plasma system exhibits more complex coupling structure and the corresponding analysis of the large time behavior of solutions becomes more complicated, referring to [3] and [5]. Hence it is meaningful and interesting to study the general physical situation for the nonlinear stability of superposition of boundary layer and rarefaction wave on the two-fluid NSP system in the future.

Finally, we refer readers to [3, 5, 6, 12, 20] and references therein for the study of the related works on the NSP system. Here we would still mention several most closely related papers: [11, 19] for the spectral analysis and time-decay of the NSP system around the constant states, [2, 18] for the global existence

of strong solutions to the one-dimensional NSP system with large data. Recently, the stability of the superposition of rarefaction wave and contact discontinuity for the NSP system with free boundary has been obtained by Ruan-Yin-Zhu [17]. For the investigations in the stability of the rarefaction wave of the related models, see also [4] for the study of the more complicated Vlasov-Poisson-Boltzmann system.

The rest of the paper is arranged as follows. In the main part Section 2, we give the *a priori* estimates on the solutions of the perturbative equations. The structure of Poisson equation and the symmetry of two-fluid system play important roles in the proof of the *a priori* estimates. The proof of Theorem 1.1 is concluded in Section 3.

Notations: Throughout this paper, C denotes some positive constant (generally large) and c denotes some positive constant (generally small), where both C and c may take different values in different places. $L^p = L^p(\mathbb{R}_+)$ $(1 \le p \le +\infty)$ denotes the usual Lebesgue space on $\mathbb{R}_+$ with its norm $\|\cdot\|_{L^p}$, and when $p = 2, +\infty$, we write $\|\cdot\|_{L^2(\mathbb{R}_+)} = \|\cdot\|$ and $\|\cdot\|_{L^\infty(\mathbb{R}_+)} = \|\cdot\|_\infty$. We use $H^s = H^s(\mathbb{R}_+)$ $(s \ge 0)$ to denote the usual Sobolev space with respect to x variable.

2 The proof of a priori estimates

Let $[\rho_i, u_i, \rho_e, u_e, E]$ be the solution of the one-dimensional two-fluid Navier-Stokes-Poisson system (1.1), (1.3) and (1.4). Let $[\hat\rho, \hat u]$ be the solution of (1.21). Now, we put the perturbation $[\varphi_i, \psi_i, \varphi_e, \psi_e]$ by

$$\varphi_i = \rho_i - \hat\rho, \quad \psi_i = u_i - \hat u, \quad \varphi_e = \rho_e - \hat\rho, \quad \psi_e = u_e - \hat u. \tag{2.1}$$

Then, from (1.1) and (1.21), $[\varphi_i, \psi_i, \varphi_e, \psi_e]$ satisfies

$$\begin{cases} \partial_t\varphi_i + u_i\partial_x\varphi_i + \rho_i\partial_x\psi_i = -f_i, \\ \rho_i(\partial_t\psi_i + u_i\partial_x\psi_i) + P'(\rho_i)\partial_x\varphi_i = \partial_x^2\psi_i - g_i + \rho_i E, \\ \partial_t\varphi_e + u_e\partial_x\varphi_e + \rho_e\partial_x\psi_e = -f_e, \\ \rho_e(\partial_t\psi_e + u_e\partial_x\psi_e) + P'(\rho_e)\partial_x\varphi_e = \partial_x^2\psi_e - g_e - \rho_e E, \\ \partial_x E = \varphi_i - \varphi_e, \quad x \in \mathbb{R}_+, \ t > 0, \\ (\psi_i, \psi_e)(0,t) = 0, \\ (\phi_i, \psi_i, \phi_e, \psi_e)(x,0) \to 0, \ \text{ as } \ x \to +\infty, \end{cases} \tag{2.2}$$

where f_α, g_α $(\alpha = i, e)$ are the nonlinear terms, given by

$$\begin{cases} f_\alpha = \partial_x\hat u\varphi_\alpha + \partial_x\hat\rho\psi_\alpha + \hat f, \\ g_\alpha = \rho_\alpha\partial_x\hat u\psi_\alpha + \partial_x\hat\rho\,[P'(\rho_\alpha) - P'(\hat\rho)] + [\partial_x^2\hat u - \partial_x P(\hat\rho)]\dfrac{\varphi_\alpha}{\hat\rho} + \hat g\dfrac{\rho_\alpha}{\hat\rho}. \end{cases} \tag{2.3}$$

We define the solution space $X(0,T)$ by

$$\begin{aligned} X(0,T) := \big\{&[\varphi_\alpha, \psi_\alpha, E] \in C([0,T]; H^1), \quad [\partial_x\varphi_\alpha, \partial_x E] \in L^2([0,T]; L^2), \\ &\partial_x\psi_\alpha \in L^2([0,T]; H^1), \quad \psi_\alpha(0,t) = 0, \quad \alpha = i, e, \quad \forall (x,t) \in [0,+\infty\} \times [0,T]\big\}. \end{aligned}$$

The local existence of (2.2) can be established by the standard iteration argument and hence will be skipped in the paper. To obtain the global existence part of Theorem 1.1, it suffices to prove the following Proposition 2.1 (*a priori* estimates).

Proposition 2.1. *(a priori estimates). Assume all the conditions listed in Theorem* 1.1 *hold. Let* $[\varphi_i, \psi_i, \varphi_e, \psi_e, E]$ *be a solution to the initial boundary value problem* (2.2) *on* $0 \le t \le T$ *for some positive constant* T*. There exist some positive constants* C *and* ε_1 *such that if*

$$\sup_{0\le t\le T}(\|[\varphi_i, \psi_i, \varphi_e, \psi_e](t)\|_{H^1} + \|E(t)\|) + \epsilon + \tilde\delta \le \varepsilon_1, \tag{2.4}$$

then the solution $[\varphi_i, \psi_i, \varphi_e, \psi_e, E]$ *satisfies*

$$\begin{aligned}&\sup_{0\le t\le T}\|[\varphi_i, \psi_i, \varphi_e, \psi_e, E]\|_{H^1}^2+\int_0^T\|\sqrt{\partial_x\hat{u}}[\varphi_i, \psi_i, \varphi_e, \psi_e]\|^2dt+\int_0^T\|\partial_x[\varphi_i,\varphi_e,E]\|^2+\|\partial_x\left[\psi_i,\psi_e\right]\|_{H^1}^2dt\\ \le& C\left(\|[\varphi_{i0},\psi_{i0},\varphi_{e0},\psi_{e0}]\|_{H^1}^2+\|E(0,t)\|^2\right)+C\left(\epsilon^{\frac{1}{10}}+\tilde{\delta}^{\frac{1}{9}}\right).\end{aligned} \tag{2.5}$$

Using (2.4) and the following Sobolev inequality

$$|h(x)|\le\sqrt{2}\|h\|^{\frac12}\|h_x\|^{\frac12} \text{ for } h(x)\ \in H^1(\mathbb{R}_+), \tag{2.6}$$

we have

$$\|[\varphi_i, \psi_i, \varphi_e, \psi_e]\|_\infty\le\sqrt{2}\varepsilon_1, \tag{2.7}$$

which will be repeatedly used in the following.

We prove Proposition 2.1 by elementary energy methods. Lemma 4.1 in the appendix plays a key role in the stability analysis. Before our estimates, we should point out that the general constant C below may depend on the strength of the rarefaction wave δ_r since the rarefaction wave considered here is not weak. Now, we prove Proposition 2.1 by the following three steps.

Step1: The zero-order energy estimates.

For $\alpha=i,e$, we define the function

$$\Phi_\alpha=\Phi(\rho_\alpha,\hat{\rho})=\int_{\hat{\rho}}^{\rho_\alpha}\frac{P(s)-P(\hat{\rho})}{s^2}ds$$

and $\eta_\alpha=\rho_\alpha\Phi_\alpha+\frac12\rho_\alpha\psi_\alpha^2$. Direct calculations give rise to

$$\begin{aligned}&\partial_t\eta_i+\partial_x\left[u_i\eta_i+(P(\rho_i)-P(\hat{\rho}))\psi_i-\psi_i\partial_x\psi_i\right]+\partial_x\hat{u}\left[P(\rho_i)-P(\hat{\rho})-P'(\hat{\rho})\varphi_i+\rho_i\psi_i^2\right]\\&+(\partial_x\psi_i)^2=\rho_i\psi_iE-\partial_x^2\hat{u}\frac{\varphi_i\psi_i}{\hat{\rho}}-\hat{g}\frac{\rho_i\psi_i}{\hat{\rho}}-P'(\hat{\rho})\hat{f}\frac{\varphi_i}{\hat{\rho}}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\partial_t\eta_e+\partial_x\left[u_e\eta_e+(P(\rho_e)-P(\hat{\rho}))\psi_e-\psi_e\partial_x\psi_e\right]+\partial_x\hat{u}\left[P(\rho_e)-P(\hat{\rho})-P'(\hat{\rho})\varphi_e+\rho_e\psi_e^2\right]\\&+(\partial_x\psi_e)^2=-\rho_e\psi_eE-\partial_x^2\hat{u}\frac{\varphi_e\psi_e}{\hat{\rho}}-\hat{g}\frac{\rho_e\psi_e}{\hat{\rho}}-P'(\hat{\rho})\hat{f}\frac{\varphi_e}{\hat{\rho}}.\end{aligned} \tag{2.9}$$

Taking the summation of (2.8) and (2.9), and integrating the resulting equation with respect to x over $\mathbb{R}_+$, we arrive at

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}_+}(\eta_i+\eta_e)dx+|u_b|\left[(\rho_i\Phi_i)(0,t)+(\rho_e\Phi_e)(0,t)\right]+\int_{\mathbb{R}_+}\left[(\partial_x\psi_i)^2+(\partial_x\psi_e)^2\right]dx\\&+\int_{\mathbb{R}_+}\partial_x\hat{u}\left[P(\rho_i)-P(\hat{\rho})-P'(\hat{\rho})\varphi_i+P(\rho_e)-P(\hat{\rho})-P'(\hat{\rho})\varphi_e+\rho_i\psi_i^2+\rho_e\psi_e^2\right]dx\\=&\underbrace{\int_{\mathbb{R}_+}(\rho_i\psi_i-\rho_e\psi_e)Edx}_{I_1}\underbrace{-\int_{\mathbb{R}_+}\left(\partial_x^2\hat{u}\frac{\varphi_i\psi_i}{\hat{\rho}}+\partial_x^2\hat{u}\frac{\varphi_e\psi_e}{\hat{\rho}}\right)dx}_{Q_1}\\&\underbrace{-\int_{\mathbb{R}_+}\left(\hat{g}\frac{\rho_i\psi_i}{\hat{\rho}}+\hat{g}\frac{\rho_e\psi_e}{\hat{\rho}}\right)dx}_{Q_2}\underbrace{-\int_{\mathbb{R}_+}\left(P'(\hat{\rho})\hat{f}\frac{\varphi_i}{\hat{\rho}}+P'(\hat{\rho})\hat{f}\frac{\varphi_e}{\hat{\rho}}\right)dx}_{Q_3}.\end{aligned} \tag{2.10}$$

Here we have used the boundary condition $(2.2)_6$ and $u_b<0$.

From Poisson equation and mass conservation equation, we have

$$\partial_x E = \rho_i - \rho_e, \quad \partial_t E = \rho_e u_e - \rho_i u_i. \tag{2.11}$$

Now we mainly make use of (2.11) to deal with the difficult term I_1. Then one has by integration by parts

$$\begin{aligned} I_1 =& \int_{\mathbb{R}_+} E(\rho_i u_i - \rho_e u_e)dx - \int_{\mathbb{R}_+} E(\rho_i - \rho_e)\tilde{u}dx \\ =& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+} E^2 dx - \frac{|u_b|}{2}E^2(0,t) + \underbrace{\frac{1}{2}\int_{\mathbb{R}_+} \partial_x \hat{u} E^2 dx}_{I_2}. \end{aligned} \tag{2.12}$$

Notice that $\partial_x \hat{u} = \partial_x \tilde{u} + \partial_x u^{r_2} \geq 0$ from $\partial_x \tilde{u} \geq 0$ and $\partial_x u^{r_2} \geq 0$. Now we pay our attention on the bad term I_2 since the electric field E is no longer L^2 integrate in space and time due to the structure of the Poisson equation. The main idea is to make use of the good term

$$\int_{\mathbb{R}_+} \partial_x \hat{u} \left[\rho_i \psi_i^2 + \rho_e \psi_e^2\right] dx$$

to absorb I_2. For this, multiplying $(2.2)_2$ and $(2.2)_4$ by $\frac{1}{4\rho_i} E\partial_x \hat{u}$ and $-\frac{1}{4\rho_e} E\partial_x \hat{u}$ respectively, then integrating the resulting equations over $\mathbb{R}_+$ and taking the summation of the resulting equations, one has

$$\begin{aligned} I_2 =& \frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}_+} \partial_x \hat{u}(\psi_i - \psi_e)E dx \underbrace{-\frac{1}{4}\int_{\mathbb{R}_+} (\psi_i - \psi_e)\,\partial_t E \partial_x \hat{u} dx}_{I_3} \underbrace{-\frac{1}{4}\int_{\mathbb{R}_+} (\psi_i - \psi_e)\, E \partial_t \partial_x \hat{u} dx}_{I_4} \\ & \underbrace{+\frac{1}{4}\int_{\mathbb{R}_+} (u_i \partial_x \psi_i - u_e \partial_x \psi_e)\, E\partial_x \hat{u} dx}_{I_5} \underbrace{-\frac{1}{4}\int_{\mathbb{R}_+} \left(\frac{\partial_x^2 \psi_i}{\rho_i} - \frac{\partial_x^2 \psi_e}{\rho_e}\right) E\partial_x \hat{u} dx}_{I_6} \\ & \underbrace{+\frac{1}{4}\int_{\mathbb{R}_+} \left(\frac{P'(\rho_i)}{\rho_i}\partial_x \varphi_i - \frac{P'(\rho_e)}{\rho_e}\partial_x \varphi_e\right) E\partial_x \hat{u} dx}_{I_7} \underbrace{+\frac{1}{4}\int_{\mathbb{R}_+} \left(\frac{g_i}{\rho_i} - \frac{g_e}{\rho_e}\right) E\partial_x \hat{u} dx}_{I_8}. \end{aligned} \tag{2.13}$$

From (2.11), we have

$$\partial_t E = \rho_e \psi_e - \rho_i \psi_i + (\varphi_e - \varphi_i)\hat{u}. \tag{2.14}$$

Then we make use of (2.14) to deal with the difficult term I_3. Therefore, one has

$$I_3 = \underbrace{\frac{1}{4}\int_{\mathbb{R}_+} \partial_x \hat{u}(\rho_i \psi_i^2 + \rho_e \psi_e^2)dx}_{I_9} \underbrace{-\frac{1}{4}\int_{\mathbb{R}_+} \partial_x \hat{u}(\rho_e + \rho_i)\psi_i \psi_e dx}_{I_{10}} \underbrace{+\frac{1}{4}\int_{\mathbb{R}_+} \partial_x \hat{u}(\psi_e - \psi_i)(\varphi_e - \varphi_i)\hat{u} dx}_{I_{11}}. \tag{2.15}$$

Combining (2.10)-(2.15), we arrive at the following equality

$$\begin{aligned} & \frac{d}{dt}\int_{\mathbb{R}_+} \left(\eta_i + \eta_e + \frac{E^2}{2}\right) dx - \frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}_+} \partial_x \hat{u}(\psi_i - \psi_e)E dx + |u_b|\left[\rho_i \Phi_i(0,t) + \rho_e \Phi_e(0,t) + \frac{E^2}{2}(0,t)\right] \\ & + \int_{\mathbb{R}_+} \left[(\partial_x \psi_i)^2 + (\partial_x \psi_e)^2\right] dx + \int_{\mathbb{R}_+} \partial_x \hat{u}\left[P(\rho_i) - P(\hat{\rho}) - P'(\hat{\rho})\varphi_i + P(\rho_e) - P(\hat{\rho}) - P'(\hat{\rho})\varphi_e\right] dx \\ & + \left[\int_{\mathbb{R}_+} \partial_x \hat{u}\left[\rho_i \psi_i^2 + \rho_e \psi_e^2\right] dx - I_9 - I_{10}\right] = \sum_{i=1}^{3} Q_i + \sum_{i=4}^{8} I_i + I_{11}. \end{aligned} \tag{2.16}$$

First of all, we use (2.7) to deal with the left terms in (2.16) as follows:

$$|u_b|\left[\rho_i\Phi_i(0,t)+\rho_e\Phi_e(0,t)+\frac{E^2}{2}(0,t)\right]\geq c\left[\varphi_i^2(0,t)+\varphi_e^2(0,t)+E^2(0,t)\right],$$

$$\int_{\mathbb{R}_+}\partial_x\hat{u}\left[P(\rho_i)-P(\hat{\rho})-P'(\hat{\rho})\varphi_i+P(\rho_e)-P(\hat{\rho})-P'(\hat{\rho})\varphi_e\right]dx\geq c\|\sqrt{\partial_x\hat{u}}[\varphi_i,\varphi_e]\|^2,$$

and

$$\begin{aligned}&\int_{\mathbb{R}_+}\partial_x\hat{u}\left[\rho_i\psi_i^2+\rho_e\psi_e^2\right]dx-I_9-I_{10}=\int_{\mathbb{R}_+}\partial_x\hat{u}\left[\frac{3}{4}\rho_i\psi_i^2+\frac{1}{4}\left(\rho_e+\rho_i\right)\psi_i\psi_e+\frac{3}{4}\rho_e\psi_e^2\right]dx\\ \geq&\frac{1}{4}\int_{\mathbb{R}_+}\hat{\rho}\partial_x\hat{u}\left[3\psi_i^2+2\psi_i\psi_e+3\psi_e^2\right]dx-C\|[\varphi_i,\varphi_e]\|_\infty\|\sqrt{\partial_x\hat{u}}[\psi_i,\psi_e]\|^2\\ \geq&\frac{1}{4}\int_{\mathbb{R}_+}\hat{\rho}\partial_x\hat{u}\left[2(\psi_i^2+\psi_e^2)+(\psi_i+\psi_e)^2\right]dx-C\varepsilon_1\|\sqrt{\partial_x\hat{u}}[\psi_i,\psi_e]\|^2.\end{aligned}$$

Therefore, we have

$$\int_{\mathbb{R}_+}\partial_x\hat{u}\left[\rho_i\psi_i^2+\rho_e\psi_e^2\right]dx-I_9-I_{10}\geq c\|\sqrt{\partial_x\hat{u}}[\psi_i,\psi_e]\|^2,$$

where we take ε_1 small enough.

Before our estimates, we take $q=10$ and $\theta=\frac{1}{8}$ in the following for brevity. By employing (2.7), (2.4), $(2.3)_2$, $(2.2)_5$, Lemma 1.3, Lemma 4.1, Young inequality, Cauchy-Schwarz's inequality with $0<\eta<1$, Sobolev inequality (2.6), the boundary condition $\psi_i(0,t)=\psi_e(0,t)=0$ and integrating by parts, we obtain the estimates on the right terms in (2.16) as follows:

$$\begin{aligned}&|Q_1|+|Q_2|+|Q_3|\\ \leq&C\|[\varphi_i,\varphi_e,\psi_i,\psi_e]\|_\infty\int_{\mathbb{R}_+}\left(|\hat{f}|+|\hat{g}+\partial_x^2u^{r_2}|+|\partial_x^2u^{r_2}|\right)dx+C\int_{\mathbb{R}_+}|\partial_x^2\tilde{u}|\left(\varphi_i^2+\varphi_e^2+\psi_i^2+\psi_e^2\right)dx\\ \leq&C\|[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^{\frac{1}{2}}\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^{\frac{1}{2}}\left[\frac{\tilde{\delta}}{1+\tilde{\delta}t}+\epsilon^\theta(1+t)^{-(1-\theta)}\ln(1+\tilde{\delta}t)+\epsilon^{\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}\right]\\ &+C\tilde{\delta}^2\left[\varphi_i^2(0,t)+\varphi_e^2(0,t)\right]+C\tilde{\delta}\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2\\ \leq&C(\tilde{\delta}^{\frac{2}{3}}+\epsilon^{\frac{1}{10}})\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2+C\frac{\tilde{\delta}^{\frac{10}{9}}}{(1+\tilde{\delta}t)^{\frac{4}{3}}}+C\epsilon^{\frac{1}{10}}(1+t)^{-\frac{13}{12}}+C\tilde{\delta}^2\left[\varphi_i^2(0,t)+\varphi_e^2(0,t)\right],\end{aligned}$$

$$\begin{aligned}|I_4|\leq&C\|[\psi_i,\psi_e]\|_\infty\|E\|_\infty\|[\partial_t\partial_xu^{r_2}]\|_{L^1}\\ \leq&C\epsilon^{\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}\|[\psi_i,\psi_e]\|^{\frac{1}{2}}\|\partial_x[\psi_i,\psi_e]\|^{\frac{1}{2}}\|E\|^{\frac{1}{2}}\|\partial_xE\|^{\frac{1}{2}}\\ \leq&C\epsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{5}}+C\epsilon^{\frac{1}{10}}\|\partial_x[\psi_i,\psi_e,E]\|^2,\end{aligned}$$

$$\begin{aligned}&|I_5|+|I_6|+|I_7\\ \leq&C\|\partial_xu^{r_2}\|_\infty\|E\|\|\partial_x[\psi_i,\psi_e,\partial_x\psi_i,\partial_x\psi_e,\varphi_i,\varphi_e]\|\\ &+C\int_{\mathbb{R}_+}|\partial_x[\psi_i,\psi_e,\partial_x\psi_i,\partial_x\psi_e,\varphi_i,\varphi_e]|\,|E|\partial_x\tilde{u}dx\\ \leq&C\epsilon^\theta(1+t)^{-(1-\theta)}\|E\|\|\partial_x[\psi_i,\psi_e,\partial_x\psi_i,\partial_x\psi_e,\varphi_i,\varphi_e]\|\\ &+C\tilde{\delta}\|\partial_x[\psi_i,\psi_e,\partial_x\psi_i,\partial_x\psi_e,\varphi_i,\varphi_e,E]\|^2+C\tilde{\delta}^2E^2(0,t)\\ \leq&C(\tilde{\delta}+\epsilon^{\frac{1}{8}})\|\partial_x[\psi_i,\psi_e,\partial_x\psi_i,\partial_x\psi_e,\varphi_i,\varphi_e,E]\|^2+C\tilde{\delta}^2E^2(0,t)+C\epsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{4}},\end{aligned}$$

$$
\begin{aligned}
|I_8| \leq & C\int_{\mathbb{R}_+}(|g_i|+|g_e|)\,|E||\partial_x\hat{u}|dx\\
\leq & C\int_{\mathbb{R}_+}\left\{|\partial_x\hat{u}|(|\psi_i|+|\psi_e|+|\varphi_i|+|\varphi_e|)+|\partial_x^2\hat{u}|(|\varphi_i|+|\varphi_e|)\right\}|E||\partial_x\hat{u}|dx+C\int_{\mathbb{R}_+}|\hat{g}||E||\partial_x\hat{u}|dx\\
\leq & C\int_{\mathbb{R}_+}\left(|\partial_x\tilde{u}|^2+|\partial_x^2\tilde{u}|+|\partial_x u^{r_2}|^2+|\partial_x^2u^{r_2}|\right)\left(\psi_i^2+\psi_e^2+\varphi_i^2+\varphi_e^2+E^2\right)dx+C\int_{\mathbb{R}_+}|\hat{g}|^2dx\\
\leq & C\tilde{\delta}^2\left[\varphi_i^2(0,t)+\varphi_e^2(0,t)+E^2(0,t)\right]+C\tilde{\delta}\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e,E]\|^2+C\epsilon^{1+\frac{2}{q}}(1+t)^{-2(1-\frac{1}{q})}+C\tilde{\delta}(1+t)^{-2}\\
&+C\left(\|[\varphi_i,\varphi_e,\psi_i,\psi_e,E]\|\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e,E]\|\right)\left(\|\partial_x^2u^{r_2}\|_{L^1}+\|\partial_x u^{r_2}\|^2\right)\\
\leq & C(\tilde{\delta}+\epsilon^{\frac{1}{10}})\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e,E]\|^2+C(\epsilon^{\frac{1}{10}}+\tilde{\delta})(1+t)^{-\frac{9}{5}}+C\tilde{\delta}^2\left[\varphi_i^2(0,t)+\varphi_e^2(0,t)+E^2(0,t)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{11}| \leq & C\int_{\mathbb{R}_+}|\partial_x\tilde{u}|\left(|\psi_i|+|\psi_e|\right)|\partial_x E|dx+C\int_{\mathbb{R}_+}|\partial_x u^{r_2}|\left(|\psi_i|+|\psi_e|\right)|\partial_x E|dx\\
\leq & \eta\|\partial_x E\|^2+C_\eta\int_{\mathbb{R}_+}|\partial_x\tilde{u}|^2\left(|\psi_i|^2+|\psi_e|^2\right)dx+C\|\partial_x u^{r_2}\|_\infty(\|\psi_i\|+\|\psi_e\|)\|\partial_x E\|\\
\leq & \eta\|\partial_x E\|^2+C_\eta\tilde{\delta}^2\|\partial_x[\psi_i,\psi_e]\|^2+C\epsilon^\theta(1+t)^{-2(1-\theta)}(\|\psi_i\|^2+\|\psi_e\|^2)+C\epsilon^\theta\|\partial_x E\|^2\\
\leq & (\eta+C\epsilon^{\frac{1}{8}})\|\partial_x E\|^2+C_\eta\tilde{\delta}^2\|\partial_x[\psi_i,\psi_e]\|^2+C\epsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{4}}.
\end{aligned}
$$

Substituting the estimates above into (2.16) and integrating the resulting inequality over $[0,T]$ and using Cauchy Schwarz's inequality, and taking ϵ, $\tilde{\delta}$ and ε_1 small enough, one can see that

$$
\begin{aligned}
&\|[\varphi_i,\varphi_e,\psi_i,\psi_e,E]\|^2+\int_0^T\left[\|\partial_x[\psi_i,\psi_e]\|^2+\|\sqrt{\partial_x\hat{u}}[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2\right]dt \qquad (2.17)\\
&+\int_0^T\left[(\varphi_i)^2(0,t)+(\varphi_e)^2(0,t)+E^2(0,t)\right]dt\\
\leq & C\left(\|[\varphi_{i0},\varphi_{e0},\psi_{i0},\psi_{e0}]\|^2+\|E(x,0)\|^2\right)+(\eta+C\epsilon^{\frac{1}{10}}+C\tilde{\delta}^{\frac{2}{3}})\|\partial_x[\varphi_i,\varphi_e,\partial_x\psi_i,\partial_x\psi_e,E]\|^2+C(\epsilon^{\frac{1}{10}}+\tilde{\delta}^{\frac{1}{9}}).
\end{aligned}
$$

Step 2. *Dissipation of* $\partial_x[\varphi_i,\varphi_e,E]$.

We first differentiate $(2.2)_1$ and $(2.2)_3$ with respect to x, respectively, to obtain

$$
\partial_t\partial_x\varphi_i+\partial_x u_i\partial_x\varphi_i+u_i\partial_x^2\varphi_i+\partial_x\rho_i\partial_x\psi_i+\rho_i\partial_x^2\psi_i+\partial_x^2\hat{u}\varphi_i+\partial_x\hat{u}\partial_x\varphi_i+\partial_x\hat{\rho}\partial_x\psi_i+\partial_x^2\hat{\rho}\psi_i+\partial_x\hat{f}=0 \quad (2.18)
$$

and

$$
\partial_t\partial_x\varphi_e+\partial_x u_e\partial_x\varphi_e+u_e\partial_x^2\varphi_e+\partial_x\rho_e\partial_x\psi_e+\rho_e\partial_x^2\psi_e+\partial_x^2\hat{u}\varphi_e+\partial_x\hat{u}\partial_x\varphi_e+\partial_x\hat{\rho}\partial_x\psi_e+\partial_x^2\hat{\rho}\psi_e+\partial_x\hat{f}=0. \quad (2.19)
$$

Then multiplying $(2.2)_5$, $(2.2)_2$, $(2.2)_4$, (2.18) and (2.19) by $\partial_x E$, $\frac{\partial_x\varphi_i}{\rho_i}$, $\frac{\partial_x\varphi_e}{\rho_e}$, $\frac{\partial_x\varphi_i}{\rho_i^2}$ and $\frac{\partial_x\varphi_e}{\rho_e^2}$, and integrating the resulting equalities over $\mathbb{R}_+$, one has

$$
\int_{\mathbb{R}_+}(\partial_x E)^2dx=[\varphi_e(0,t)-\varphi_i(0,t)]E(0,t)-\int_{\mathbb{R}_+}\partial_x(\varphi_i-\varphi_e)Edx,
$$

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\partial_t\psi_i\partial_x\varphi_i dx+\int_{\mathbb{R}_+}u_i\partial_x\psi_i\partial_x\varphi_i dx+\int_{\mathbb{R}_+}\frac{P'(\rho_i)}{\rho_i}(\partial_x\varphi_i)^2dx\\
&=\int_{\mathbb{R}_+}\partial_x\varphi_i Edx+\int_{\mathbb{R}_+}\partial_x^2\psi_i\frac{\partial_x\varphi_i}{\rho_i}dx-\int_{\mathbb{R}_+}g_i\frac{\partial_x\varphi_i}{\rho_i}dx,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\partial_t\psi_e\partial_x\varphi_e dx+\int_{\mathbb{R}_+}u_e\partial_x\psi_e\partial_x\varphi_e dx+\int_{\mathbb{R}_+}\frac{P'(\rho_e)}{\rho_e}(\partial_x\varphi_e)^2dx\\
&=-\int_{\mathbb{R}_+}\partial_x\varphi_e Edx+\int_{\mathbb{R}_+}\partial_x^2\psi_e\frac{\partial_x\varphi_e}{\rho_e}dx-\int_{\mathbb{R}_+}g_e\frac{\partial_x\varphi_e}{\rho_e}dx,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\frac{\partial_x\varphi_i}{\rho_i^2}\partial_t\partial_x\varphi_i dx+\int_{\mathbb{R}_+}\partial_x u_i\frac{(\partial_x\varphi_i)^2}{\rho_i^2}dx+\int_{\mathbb{R}_+}u_i\frac{\partial_x\varphi_i\partial_x^2\varphi_i}{\rho_i^2}dx\\
&\quad+\int_{\mathbb{R}_+}\frac{\partial_x\varphi_i}{\rho_i^2}\partial_x\rho_i\partial_x\psi_i dx+\int_{\mathbb{R}_+}\partial_x\hat{u}\frac{(\partial_x\varphi_i)^2}{\rho_i^2}dx\\
&\quad=-\int_{\mathbb{R}_+}\partial_x^2\psi_i\frac{\partial_x\varphi_i}{\rho_i}dx-\int_{\mathbb{R}_+}\partial_x^2\hat{u}\varphi_i\frac{\partial_x\varphi_i}{\rho_i^2}dx-\int_{\mathbb{R}_+}\partial_x\hat{\rho}\partial_x\psi_i\frac{\partial_x\varphi_i}{\rho_i^2}dx\\
&\quad-\int_{\mathbb{R}_+}\partial_x^2\hat{\rho}\psi_i\frac{\partial_x\varphi_i}{\rho_i^2}dx-\int_{\mathbb{R}_+}\partial_x\hat{f}\frac{\partial_x\varphi_i}{\rho_i^2}dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\frac{\partial_x\varphi_e}{\rho_e^2}\partial_t\partial_x\varphi_e dx+\int_{\mathbb{R}_+}\partial_x u_e\frac{(\partial_x\varphi_e)^2}{\rho_e^2}dx+\int_{\mathbb{R}_+}u_e\frac{\partial_x\varphi_e\partial_x^2\varphi_e}{\rho_e^2}dx\\
&\quad+\int_{\mathbb{R}_+}\frac{\partial_x\varphi_e}{\rho_e^2}\partial_x\rho_e\partial_x\psi_e dx+\int_{\mathbb{R}_+}\partial_x\hat{u}\frac{(\partial_x\varphi_e)^2}{\rho_e^2}dx\\
&\quad=-\int_{\mathbb{R}_+}\partial_x^2\psi_e\frac{\partial_x\varphi_e}{\rho_e}dx-\int_{\mathbb{R}_+}\partial_x^2\hat{u}\varphi_e\frac{\partial_x\varphi_e}{\rho_e^2}dx-\int_{\mathbb{R}_+}\partial_x\hat{\rho}\partial_x\psi_e\frac{\partial_x\varphi_e}{\rho_e^2}dx\\
&\quad-\int_{\mathbb{R}_+}\partial_x^2\hat{\rho}\psi_e\frac{\partial_x\varphi_e}{\rho_e^2}dx-\int_{\mathbb{R}_+}\partial_x\hat{f}\frac{\partial_x\varphi_e}{\rho_e^2}dx.
\end{aligned}
$$

The summation of the equalities above further implies

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}_+}(\psi_i\partial_x\varphi_i+\psi_e\partial_x\varphi_e)\,dx+\frac{d}{dt}\int_{\mathbb{R}_+}\left(\frac{1}{2\rho_i^2}(\partial_x\varphi_i)^2+\frac{1}{2\rho_e^2}(\partial_x\varphi_e)^2\right)dx\\
&\quad+\int_{\mathbb{R}_+}\left[\partial_x\hat{u}\frac{(\partial_x\varphi_i)^2}{\rho_i^2}+\partial_x\hat{u}\frac{(\partial_x\varphi_e)^2}{\rho_e^2}+\frac{P'(\rho_i)}{\rho_i}(\partial_x\varphi_i)^2+\frac{P'(\rho_e)}{\rho_e}(\partial_x\varphi_e)^2+(\partial_x E)^2\right]dx\\
&=[\varphi_e(0,t)-\varphi_i(0,t)]E(0,t)+\int_{\mathbb{R}_+}(\psi_i\partial_t\partial_x\varphi_i+\psi_e\partial_t\partial_x\varphi_e)\,dx\\
&\quad-\int_{\mathbb{R}_+}\left((\partial_x\varphi_i)^2\rho_i^{-3}\partial_t\rho_i+(\partial_x\varphi_e)^2\rho_e^{-3}\partial_t\rho_e\right)dx-\int_{\mathbb{R}_+}(u_i\partial_x\psi_i\partial_x\varphi_i+u_e\partial_x\psi_e\partial_x\varphi_e)\,dx\\
&\quad-\int_{\mathbb{R}_+}\left(g_i\frac{\partial_x\varphi_i}{\rho_i}+g_e\frac{\partial_x\varphi_e}{\rho_e}\right)dx-\int_{\mathbb{R}_+}\left(\partial_x u_i\frac{(\partial_x\varphi_i)^2}{\rho_i^2}+\partial_x u_e\frac{(\partial_x\varphi_e)^2}{\rho_e^2}\right)dx\\
&\quad-\int_{\mathbb{R}_+}\left(u_i\frac{\partial_x\varphi_i\partial_x^2\varphi_i}{\rho_i^2}+u_e\frac{\partial_x\varphi_e\partial_x^2\varphi_e}{\rho_e^2}\right)dx-\int_{\mathbb{R}_+}\left(\frac{\partial_x\varphi_i}{\rho_i^2}\partial_x\rho_i\partial_x\psi_i+\frac{\partial_x\varphi_e}{\rho_e^2}\partial_x\rho_e\partial_x\psi_e\right)dx\\
&\quad-\int_{\mathbb{R}_+}\left(\partial_x^2\hat{u}\varphi_i\frac{\partial_x\varphi_i}{\rho_i^2}+\partial_x^2\hat{u}\varphi_e\frac{\partial_x\varphi_e}{\rho_e^2}\right)dx-\int_{\mathbb{R}_+}\left(\partial_x\hat{\rho}\partial_x\psi_i\frac{\partial_x\varphi_i}{\rho_i^2}+\partial_x\hat{\rho}\partial_x\psi_e\frac{\partial_x\varphi_e}{\rho_e^2}\right)dx\\
&\quad-\int_{\mathbb{R}_+}\left(\partial_x^2\hat{\rho}\psi_i\frac{\partial_x\varphi_i}{\rho_i^2}+\partial_x^2\hat{\rho}\psi_e\frac{\partial_x\varphi_e}{\rho_e^2}\right)dx-\int_{\mathbb{R}_+}\left(\partial_x\hat{f}\frac{\partial_x\varphi_i}{\rho_i^2}+\partial_x\hat{f}\frac{\partial_x\varphi_e}{\rho_e^2}\right)dx=\sum_{l=1}^{12}J_l,
\end{aligned}
\tag{2.20}
$$

where J_l $(1\le l\le 12)$ denote the corresponding terms on the left hand side of (2.20).

Notice the fact that $|\partial_x\hat{n}|\le C\partial_x\hat{u}$, $|\partial_x^2\hat{n}|\le C(|\partial_x^2\hat{u}|+|\partial_x\hat{u}|^2)$ and $u_b<0$. We now turn to estimate J_l $(1\le l\le 12)$ term by term. By applying Holder inequality, Cauchy-Schwarz's inequality with $0<\eta<1$, Sobolev inequality (2.6), Lemma 1.3, Lemma 4.1, (2.4), (2.7), $(1.1)_1$, $(1.1)_3$, $(1.21)_1$, (1.14), (2.3), the boundary condition $\psi_i(0,t)=\psi_e(0,t)=0$, and integrating by parts, it is direct to derive the following estimates:

$$
|J_1|\le\varphi_i^2(0,t)+\varphi_e^2(0,t)+E^2(0,t),
$$

$$
\begin{aligned}
J_2 =& \int_{\mathbb{R}_+} \partial_x\psi_i\partial_x(\rho_i u_i - \hat\rho\hat u)dx + \int_{\mathbb{R}_+} \partial_x\psi_e\partial_x(\rho_e u_e - \hat\rho\hat u)dx - \int_{\mathbb{R}_+} (\psi_i+\psi_e)\partial_x\hat f dx \\
=& \int_{\mathbb{R}_+} \hat\rho[(\partial_x\psi_i)^2 + (\partial_x\psi_e)^2]dx + \int_{\mathbb{R}_+} \partial_x\hat\rho(\psi_i\partial_x\psi_i + \psi_e\partial_x\psi_e)dx \\
&+ \int_{\mathbb{R}_+} [(\partial_x\psi_i)^2\varphi_i + (\partial_x\psi_e)^2\varphi_e]dx + \int_{\mathbb{R}_+} (\varphi_i\partial_x\hat u\partial_x\psi_i + \varphi_e\partial_x\hat u\partial_x\psi_e)dx \\
&+ \int_{\mathbb{R}_+} (u_i\partial_x\psi_i\partial_x\varphi_i + u_e\partial_x\psi_e\partial_x\varphi_e)dx - \int_{\mathbb{R}_+} (\psi_i+\psi_e)\partial_x\hat f dx \\
\le& \eta\|\partial_x[\varphi_i,\varphi_e]\|^2 + C_\eta\|\partial_x[\psi_i,\psi_e]\|^2 + C(\epsilon+\tilde\delta)\|\sqrt{\partial_x\hat u}[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2 + C\|[\psi_i,\psi_e]\|_\infty\|\partial_x\hat f\|_{L^1} \\
\le& \eta\|\partial_x[\varphi_i,\varphi_e]\|^2 + C_\eta\|\partial_x[\psi_i,\psi_e]\|^2 + C(\epsilon+\tilde\delta)\|\sqrt{\partial_x\hat u}[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2 \\
&+ C\|[\psi_i,\psi_e]\|^{\frac12}\|\partial_x[\psi_i,\psi_e]\|^{\frac12}\left[\tilde\delta(1+t)^{-1} + \epsilon^\theta(1+t)^{-(1-\theta)} + \epsilon^{\frac1q}(1+t)^{-1+\frac1q}\right] \\
\le& \eta\|\partial_x[\varphi_i,\varphi_e]\|^2 + C_\eta\|\partial_x[\psi_i,\psi_e]\|^2 + C(\epsilon+\tilde\delta)\|\sqrt{\partial_x\hat u}[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2 + C(\tilde\delta^{\frac43} + \epsilon^{\frac{2}{15}})(1+t)^{-\frac76},
\end{aligned}
$$

$$
\begin{aligned}
&J_3 + J_6 + J_7 \\
=& -\frac12\left[\frac{|u_b|}{\rho_i^2(0,t)}(\partial_x\varphi_i)^2(0,t) + \frac{|u_b|}{\rho_e^2(0,t)}(\partial_x\varphi_e)^2(0,t)\right] + \int_{\mathbb{R}_+}\left(\frac12\partial_x u_i(\partial_x\varphi_i)^2\rho_i^{-2} + \frac12\partial_x u_e(\partial_x\varphi_e)^2\rho_e^{-2}\right)dx \\
\le& \int_{\mathbb{R}_+}\left(\frac12\partial_x\hat u(\partial_x\varphi_i)^2\rho_i^{-2} + \frac12\partial_x\hat u(\partial_x\varphi_e)^2\rho_e^{-2}\right)dx + \int_{\mathbb{R}_+}\left(\frac12\partial_x\psi_i(\partial_x\varphi_i)^2\rho_i^{-2} + \frac12\partial_x\psi_e(\partial_x\varphi_e)^2\rho_e^{-2}\right)dx \\
\le& C(\epsilon+\tilde\delta)\|\partial_x[\varphi_i,\varphi_e]\|^2 + C\|\partial_x[\psi_i,\psi_e]\|^{\frac12}\|\partial_x^2[\psi_i,\psi_e]\|^{\frac12}\|\partial_x[\varphi_i,\varphi_e]\|^2 \\
\le& C(\epsilon+\tilde\delta+\varepsilon_1)(\|\partial_x[\varphi_i,\varphi_e]\|^2) + C\varepsilon_1\|\partial_x^2[\psi_i,\psi_e]\|^2,
\end{aligned}
$$

$$
|J_4| + |J_{10}| \le (\eta + C\tilde\delta + C\epsilon)\|\partial_x[\varphi_i,\varphi_e]\|^2 + (C_\eta + C\tilde\delta + C\epsilon)\|\partial_x[\psi_i,\psi_e]\|^2,
$$

$$
\begin{aligned}
|J_5| \le& C\int_{\mathbb{R}_+}(|\partial_x^2\hat u| + |\partial_x\hat\rho| + \partial_x\hat u)|[\varphi_i,\varphi_e,\psi_i,\psi_e]||\partial_x[\varphi_i,\varphi_e]|dx + C\|\partial_x[\varphi_i,\varphi_e]\|\|\hat g\| \\
\le& \eta\|\partial_x[\varphi_i,\varphi_e]\|^2 + C_\eta(\|\hat g\|^2 + \|\partial_x^2 u^{r_2}\|_\infty^2 + \|\partial_x u^{r_2}\|_\infty^2) + C\tilde\delta\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2 + C\tilde\delta^2[\varphi_i^2+\varphi_e^2](0,t) \\
\le& (\eta + C\tilde\delta)\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2 + C\tilde\delta^2[\varphi_i^2(0,t)+\varphi_e^2(0,t)] + C_\eta\left[\epsilon^{2\theta}(1+t)^{-2(1-\frac1q)} + \tilde\delta(1+t)^{-2}\right] \\
\le& (\eta + C\tilde\delta)\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2 + C\tilde\delta^2[\varphi_i^2(0,t)+\varphi_e^2(0,t)] + C_\eta(\tilde\delta + \epsilon^{\frac14})(1+t)^{-\frac95},
\end{aligned}
$$

$$
\begin{aligned}
|J_8| \le& C\|\partial_x\hat\rho\|_\infty\|\partial_x[\varphi_i,\varphi_e]\|\|\partial_x[\psi_i,\psi_e]\| + C\|\partial_x[\psi_i,\psi_e]\|^{\frac12}\|\partial_x^2[\psi_i,\psi_e]\|^{\frac12}\|\partial_x[\varphi_i,\varphi_e]\|^2 \\
\le& C(\epsilon+\tilde\delta+\varepsilon_1)\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2 + C\varepsilon_1\|\partial_x^2[\psi_i,\psi_e]\|^2,
\end{aligned}
$$

$$
\begin{aligned}
|J_9| + |J_{11}| \le& C\int_{\mathbb{R}_+}(|\partial_x^2\tilde u| + |\partial_x\tilde u|^2)|[\varphi_i,\varphi_e,\psi_i,\psi_e]||\partial_x[\varphi_i,\varphi_e]|dx \\
&+ C\int_{\mathbb{R}_+}\left[|\partial_x^2 u^{r_2}| + |\partial_x u^{r_2}|^2\right]|[\varphi_i,\varphi_e,\psi_i,\psi_e]||\partial_x[\varphi_i,\varphi_e]|dx \\
\le& C\tilde\delta^4[\varphi_i^2(0,t)+\varphi_e^2(0,t)] + C\tilde\delta\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2 \\
&+ C\left[\epsilon^{1+\frac1q}(1+t)^{-1+\frac1q} + \epsilon^{2\theta}(1+t)^{-2(1-\theta)}\right]\|[\psi_i,\psi_e]\|\|\partial_x[\varphi_i,\varphi_e]\| \\
\le& C(\tilde\delta + \epsilon^{\frac14})\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2 + C\epsilon^{\frac14}(1+t)^{-\frac95},
\end{aligned}
$$

and

$$
\begin{aligned}
|J_{12}| \le& \|\partial_x[\varphi_i,\varphi_e]\|\|\partial_x\hat f\| \le \eta\|\partial_x[\varphi_i,\varphi_e]\|^2 + C_\eta\|\partial_x\hat f\|^2 \\
\le& \eta\|\partial_x[\varphi_i,\varphi_e]\|^2 + C_\eta\epsilon^{1+\frac2q}(1+t)^{-2(1-\frac1q)} + C_\eta(\tilde\delta+\epsilon)(1+t)^{-2} \\
\le& \eta\|\partial_x[\varphi_i,\varphi_e]\|^2 + C_\eta(\tilde\delta+\epsilon)(1+t)^{-\frac95},
\end{aligned}
$$

where we take $q = 10$ and $\theta = \frac18$ in the above estimates.

Inserting the above estimations for J_l $(1 \leq l \leq 12)$ into (2.20) and then choosing ε_1, ϵ, $\tilde{\delta}$ and η so small, and integrating (2.20) over $[0,T]$ and using (2.17), Cauchy-Schwarz's inequality with $0<\eta<1$, one can see that

$$\begin{aligned}
&\|\partial_x[\varphi_i,\varphi_e]\|^2+\int_0^T\|\sqrt{\partial_x\hat{u}}\partial_x[\varphi_i,\varphi_e]\|^2dt+\int_0^T\|\partial_x[\varphi_i,\varphi_e,E]\|^2dt\\
\leq& C\left(\|[\psi_{i0},\psi_{e0}]\|^2+\|E(x,0)\|^2+\|[\varphi_{i0},\varphi_{e0}]\|^2_{H^1}\right)+(\eta+C\epsilon^{\frac{1}{10}}+C\tilde{\delta}^{\frac{2}{3}}+\varepsilon_1)\|\partial_x^2[\psi_i,\psi_e]\|^2+C(\epsilon^{\frac{1}{10}}+\tilde{\delta}^{\frac{1}{9}}).
\end{aligned} \tag{2.21}$$

Step 3. *Dissipation of* $\partial_x^2[\psi_i,\psi_e]$.

Multiplying $(2.2)_2$ and $(2.2)_4$ by $-\frac{\partial_x^2\psi_i}{\rho_i}$ and $-\frac{\partial_x^2\psi_e}{\rho_e}$ respectively, and then integrating the resulting equations over $\mathbb{R}_+$ and taking the summation of the resulting equations, one has

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}_+}\left(\frac{1}{2}(\partial_x\psi_i)^2+\frac{1}{2}(\partial_x\psi_e)^2\right)dx+\int_{\mathbb{R}_+}\left(\frac{(\partial_x^2\psi_i)^2}{\rho_i}+\frac{(\partial_x^2\psi_e)^2}{\rho_e}\right)dx\\
=&-\int_{\mathbb{R}_+}E\partial_x^2(\psi_i-\psi_e)dx+\int_{\mathbb{R}_+}\left(\frac{P'(\rho_i)}{\rho_i}\partial_x\varphi_i\partial_x^2\psi_i+\frac{P'(\rho_e)}{\rho_e}\partial_x\varphi_e\partial_x^2\psi_e\right)dx\\
&+\int_{\mathbb{R}_+}\left(u_i\partial_x\psi_i\partial_x^2\psi_i+u_e\partial_x\psi_e\partial_x^2\psi_e\right)dx+\int_{\mathbb{R}_+}\left(\frac{g_i}{\rho_i}\partial_x^2\psi_i+\frac{g_e}{\rho_e}\partial_x^2\psi_e\right)dx\\
=&\sum_{l=13}^{16}J_l,
\end{aligned} \tag{2.22}$$

where we have used the boundary condition $\psi_i(0,t)=\psi_e(0,t)=0$.

We now turn to estimate J_l $(13 \leq J \leq 16)$ term by term. By applying Cauchy-Schwarz's inequality with $0<\eta<1$, Sobolev inequality (2.6), Lemma 1.3, Lemma 4.1, (1.14) and integrating by parts, we can obtain that

$$\begin{aligned}
J_{13}=&E(0,t)[(\partial_x\psi_i)(0,t)-(\partial_x\psi_e)(0,t)]+\int_{\mathbb{R}_+}\partial_xE\partial_x(\psi_i-\psi_e)dx\\
\leq&\eta[(\partial_x\psi_i)^2(0,t)+(\partial_x\psi_e)^2(0,t)]+C_\eta E^2(0,t)+\frac{1}{2}\|\partial_x[\psi_i,\psi_e]\|^2+\frac{1}{2}\|\partial_xE\|^2\\
\leq&\eta\|\partial_x[\psi_i,\psi_e]\|^2_\infty+C_\eta E^2(0,t)+\frac{1}{2}\|\partial_x[\psi_i,\psi_e]\|^2+\frac{1}{2}\|\partial_xE\|^2\\
\leq&\eta(\|\partial_x[\psi_i,\psi_e]\|^2+\|\partial_x^2[\psi_i,\psi_e]\|^2)+C_\eta E^2(0,t)+\frac{1}{2}\|\partial_x[\psi_i,\psi_e]\|^2+\frac{1}{2}\|\partial_xE\|^2,
\end{aligned}$$

$$|J_{14}|+|J_{15}|\leq\eta\|\partial_x^2[\psi_i,\psi_e]\|^2+C_\eta\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2$$

and

$$|J_{16}|\leq(\eta+C\tilde{\delta})\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e,\partial_x\psi_i,\partial_x\psi_e]\|^2+C\tilde{\delta}^2[\varphi_i^2(0,t)+\varphi_e^2(0,t)]+C_\eta(\tilde{\delta}+\epsilon^{\frac{1}{4}})(1+t)^{-\frac{9}{5}},$$

where we take $q=10$ in the above estimates and the estimate of J_{16} is the same as J_5.

Inserting the above estimations for J_l $(13 \leq J \leq 16)$ into (2.22) and then integrating (2.22) over $[0,T]$ and using (2.17) and (2.21), one can see that

$$\begin{aligned}
&\|\partial_x[\psi_i,\psi_e]\|^2+\int_0^T\|\partial_x^2[\psi_i,\psi_e]\|^2dt\\
\leq& C\left(\|[\varphi_{i0},\varphi_{e0},\psi_{i0},\psi_{e0}]\|^2_{H^1}+\|[E(x,0)]\|^2\right)+C\left(\epsilon^{\frac{1}{10}}+\tilde{\delta}^{\frac{1}{9}}\right).
\end{aligned} \tag{2.23}$$

where we choose ε_1, ϵ, $\tilde{\delta}$ and η sufficiently small.

Proof of Proposition 2.1. Now, following Step 1, Step 2 and Step 3, we are ready to prove Proposition 2.1. Summing up the estimates (2.17), (2.21), (2.23) and taking ϵ, $\tilde{\delta}$, ε_1, η suitably small, we have

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\|[\varphi_i,\varphi_e,\psi_i,\psi_e]\|_{H^1}^2+\|E\|^2\right)+\int_0^T\|\sqrt{\partial_x\hat{u}}[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2dt\\
&+\int_0^T\|\partial_x[\varphi_i,\varphi_e,E]\|^2dt+\int_0^T\|\partial_x\left[\psi_i,\psi_e\right]\|_{H^1}^2dt\\
\le& C\left(\|[\varphi_{i0},\varphi_{e0},\psi_{i0},\psi_{e0}]\|_{H^1}^2+\|[E(x,0)]\|^2+\epsilon^{\frac{1}{10}}+\tilde{\delta}^{\frac{1}{9}}\right).
\end{aligned}\tag{2.24}$$

From $(2.2)_5$, it follows

$$\|\partial_xE\|^2\le\|[\varphi_i,\varphi_e]\|^2,$$

this and (2.24) imply the desired estimate (2.5). Thus the proof of Proposition 2.1 is completed. □

3 Global existence and large time behavior

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By the *a priori* estimates (2.5), there exists a positive constant C_0 such that

$$\|[\varphi_i,\varphi_e,\psi_i,\psi_e,E]\|_{H^1}^2\le C_0\left(\|[\varphi_{i0},\varphi_{e0},\psi_{i0},\psi_{e0}]\|_{H^1}^2+\|[E(x,0)]\|^2+\epsilon^{\frac{1}{10}}+\tilde{\delta}^{\frac{1}{9}}\right)\tag{3.1}$$

holds. It is straightforward to see that there exists a small constant ε_0 such that if

$$\|[\varphi_{i0},\varphi_{e0},\psi_{i0},\psi_{e0}]\|_{H^1}^2+\|E(0,x)\|^2\le\varepsilon_0^2,$$

we can close the *a priori* assumption (2.4) by choosing $\varepsilon_1=4\sqrt{C_0(\varepsilon_0^2+\epsilon^{\frac{1}{10}}+\tilde{\delta}^{\frac{1}{9}})}$. By letting ϵ and $\tilde{\delta}$ be small enough, then the global existence of the solution of (2.2) follows from the standard continuation argument based on the local existence and the *a priori* estimates in Proposition 2.1. Moreover, (3.1) and (1.24) imply (1.25). Our intention next is to prove the large time behavior as (1.26) and (1.27). For this, we first justify the following limits:

$$\lim_{t\to+\infty}\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e](t)\|_{L^2}^2=0,\tag{3.2}$$

and

$$\lim_{t\to+\infty}\|\partial_xE(t)\|^2=0.\tag{3.3}$$

To prove (3.2) and (3.3), we get from (2.18), (2.19), (2.22) and (2.5) that

$$\begin{aligned}
&\int_0^{+\infty}\left|\frac{d}{dt}\|\partial_x[\varphi_i,\varphi_e,\psi_i,\psi_e]\|^2\right|dt\\
=&2\int_0^{+\infty}\left[\left|\int_{\mathbb{R}_+}\partial_t\partial_x\varphi_i\partial_x\varphi_idx\right|+\left|\int_{\mathbb{R}_+}\partial_t\partial_x\varphi_e\partial_x\varphi_edx\right|\right]dt+\int_0^{+\infty}\left|\frac{d}{dt}\|\partial_x[\psi_i,\psi_e]\|^2\right|dt\\
\le& C+C\int_0^{+\infty}\|\partial_x\left[\varphi_i,\varphi_e,\psi_i,\psi_e,E,\partial_x\left[\psi_i,\psi_e\right]\right]\|^2dt<+\infty.
\end{aligned}\tag{3.4}$$

On the other hand, $(2.2)_5$, $(2.2)_1$, $(2.2)_3$ and (2.5) yield

$$\begin{aligned}
&\int_0^{+\infty}\left|\frac{d}{dt}\|\partial_xE\|^2\right|dt=2\int_0^{+\infty}\left|\int_{\mathbb{R}_+}\partial_t\partial_xE\partial_xEdx\right|dt\\
=&2\int_0^{+\infty}\left|\int_{\mathbb{R}_+}\left(\partial_t\varphi_i-\partial_t\varphi_e\right)\partial_xEdx\right|dt<+\infty.
\end{aligned}\tag{3.5}$$

Consequently, (3.4), (3.5) together with (2.5) gives (3.2) and (3.3). Then (1.26) and (1.27) follows from (3.2), (3.3) and Sobolev's inequality (2.6). This ends the proof of Theorem 1.1. □

4 Appendix

In this appendix, we will give the following inequalities stated in Lemma 4.1 repeatedly used in the paper.

Lemma 4.1. *(i) For any function h and $(k+1)j > 2$, there is a positive constant C such that,*

$$\int_{\mathbb{R}_+} |\partial_x^k(\tilde{u} - u_*)|^j |h|^2 dx \le C\tilde{\delta}^{(k+1)j-2}\left[\tilde{\delta}h^2(0,t) + \|\partial_x h(t)\|^2\right]. \tag{4.1}$$

(ii) For any functions f, h and $2(k+1)j > 3$, there is a positive constant C such that,

$$\int_{\mathbb{R}_+} |\partial_x^k(\tilde{u} - u_*)|^j |h\partial_x f| dx \le \tilde{\delta}\|\partial_x f(t)\|^2 + C\tilde{\delta}^{2(k+1)j-3}\left[\tilde{\delta}h^2(0,t) + \|\partial_x h(t)\|^2\right]. \tag{4.2}$$

(iii) For any $\theta \in [0,1]$, we have

$$\|\partial_x(n^{r_2} - n_*), \partial_x(u^{r_2} - u_*)\|_\infty \le C\epsilon^\theta(1+t)^{-(1-\theta)}. \tag{4.3}$$

(iv) For any $\theta \in [0,1]$, $q \ge 10$, we have

$$\int_{\mathbb{R}_+} \left(|\hat{f}| + |\hat{g} + \partial_x^2 u^{r_2}|\right) dx \le C\frac{\tilde{\delta}}{1+\tilde{\delta}t} + C\epsilon^\theta(1+t)^{-(1-\theta)}\ln(1+\tilde{\delta}t) \tag{4.4}$$

and

$$\int_{\mathbb{R}_+} |\partial_x \hat{f}| dx \le C\tilde{\delta}(1+t)^{-1} + C\epsilon^\theta(1+t)^{-(1-\theta)} + C\epsilon^{\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}. \tag{4.5}$$

(v) For $q \ge 10$, we have

$$\int_{\mathbb{R}_+} |\hat{g}|^2 dx \le C\epsilon^{1+\frac{2}{q}}(1+t)^{-2(1-\frac{1}{q})} + C\tilde{\delta}(1+t)^{-2} \tag{4.6}$$

and

$$\int_{\mathbb{R}_+} |\partial_x \hat{f}|^2 dx \le C\epsilon^{1+\frac{2}{q}}(1+t)^{-2(1-\frac{1}{q})} + C(\tilde{\delta}+\epsilon)(1+t)^{-2}. \tag{4.7}$$

Proof. (i) Using (1.14) and the following Poincaré type inequalities

$$|h(x,t)| \le |h(0,t)| + x^{\frac{1}{2}}\|\partial_x h(t)\|, \tag{4.8}$$

for $(k+1)j > 2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+} |\partial_x^k(\tilde{u} - u_*)|^j |h|^2 dx \\
\le&\int_{\mathbb{R}_+} |\partial_x^k(\tilde{u} - u_*)|^j \left(h^2(0,t) + x\|\partial_x h(t)\|^2\right) dx \\
\le& Ch^2(0,t)\int_{\mathbb{R}_+} \frac{\tilde{\delta}^{(k+1)j}}{(1+\tilde{\delta}x)^{(k+1)j}} dx + C\|\partial_x h(t)\|^2 \int_{\mathbb{R}_+} \frac{x\tilde{\delta}^{(k+1)j}}{(1+\tilde{\delta}x)^{(k+1)j}} dx \\
\le& C\tilde{\delta}^{(k+1)j-2}\left[\tilde{\delta}h^2(0,t) + \|\partial_x h(t)\|^2\right].
\end{aligned}$$

(ii) By the Young inequality and Lemma 4.1 (i), for $2(k+1)j > 3$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+} |\partial_x^k(\tilde{u} - u_*)|^j |h\partial_x f| dx \\
\le&\tilde{\delta}\|\partial_x f(t)\|^2 + C\int_{\mathbb{R}_+} \frac{\tilde{\delta}^{2(k+1)j-1}}{(1+\tilde{\delta}x)^{2(k+1)j}} h^2 dx \\
\le&\tilde{\delta}\|\partial_x f(t)\|^2 + C\tilde{\delta}^{2(k+1)j-3}\left[\tilde{\delta}h^2(0,t) + \|\partial_x h(t)\|^2\right].
\end{aligned}$$

(iii) From Lemma 1.3 (ii), we have

$$\|\partial_x(n^{r_2}-n_*),\partial_x(u^{r_2}-u_*)\|_\infty \le C\min\{\epsilon,(1+t)^{-1}\}.$$

Thus we have

$$\|\partial_x(n^{r_2}-n_*),\partial_x(u^{r_2}-u_*)\|_\infty \le C\epsilon^\theta(1+t)^{-(1-\theta)}.$$

Here we have used the fact that if $0<C\le A$ and $0<C\le B$, then $C\le A^\theta B^{1-\theta}$ for any $0\le\theta\le 1$.

(iv) Using (1.23), (1.14), Lemma 1.3 (ii) and Lemma 4.1 (iii), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\left(|\hat{f}|+|\hat{g}+\partial_x^2u^{r_2}|\right)dx\\
\le& C\int_{\mathbb{R}_+}\{\partial_x\tilde{u}(u^{r_2}-u_*)+\partial_xu^{r_2}(u_*-\tilde{u})\}dx\\
=& C\int_{\mathbb{R}_+}\partial_x[(u^{r_2}-u_*)(\tilde{u}-u_*)]dx+2C\int_{\mathbb{R}_+}\partial_xu^{r_2}(u_*-\tilde{u})dx\\
=& 2C\int_0^t\partial_xu^{r_2}(u_*-\tilde{u})dx+2C\int_t^{+\infty}\partial_xu^{r_2}(u_*-\tilde{u})dx\\
\le& C\|\partial_xu^{r_2}\|_\infty\int_0^t\frac{\tilde{\delta}}{1+\tilde{\delta}x}dx+C\frac{\tilde{\delta}}{1+\tilde{\delta}t}\int_t^{+\infty}\partial_xu^{r_2}dx\\
\le& C\|\partial_xu^{r_2}\|_\infty\ln(1+\tilde{\delta}t)+C\frac{\tilde{\delta}}{1+\tilde{\delta}t}\|\partial_xu^{r_2}\|_{L^1}\\
\le& C\epsilon^\theta(1+t)^{-(1-\theta)}\ln(1+\tilde{\delta}t)+C\frac{\tilde{\delta}}{1+\tilde{\delta}t}.
\end{aligned}$$

where we have used $u^{r_2}(0,t)=u_*$ and $\tilde{u}\to u_*$ as $x\to+\infty$.

Similarly, we can obtain that

$$\begin{aligned}
\int_{\mathbb{R}_+}|\partial_x\hat{f}|dx \le& C\int_{\mathbb{R}_+}\left\{(|\partial_x^2\tilde{u}|+(\partial_x\tilde{u})^2)(u^{r_2}-u_*)+\partial_x\tilde{u}\partial_xu^{r_2}+|\partial_x^2u^{r_2}|+(\partial_xu^{r_2})^2\right\}dx\\
\le& C\|\partial_xu^{r_2}\|_\infty\int_{\mathbb{R}_+}x(|\partial_x^2\tilde{u}|+(\partial_x\tilde{u})^2)dx+C\|\partial_xu^{r_2}\|_\infty\|\partial_x\tilde{u}\|_{L^1}+C\|\partial_x^2u^{r_2}\|_{L^1}+C\|\partial_xu^{r_2}\|^2\\
\le& C\tilde{\delta}(1+t)^{-1}+C\epsilon^\theta(1+t)^{-(1-\theta)}+C\epsilon^{\frac{1}{q}}(1+t)^{-1+\frac{1}{q}},
\end{aligned}$$

where we have used the fact that $u^{r_2}(0,t)=u_*$ which yields $u^{r_2}(x,t)-u_*\le x\|\partial_xu^{r_2}\|_\infty$.

(v) Noticing (1.23) and the fact that $u^{r_2}(x,t)-u_*\le x\|\partial_xu^{r_2}\|_\infty$, and applying Lemma 1.3 and (1.14), we obtain that

$$\begin{aligned}
\int_{\mathbb{R}_+}|\hat{g}|^2dx \le& C\int_{\mathbb{R}_+}\{|\partial_x^2u^{r_2}|^2+|\partial_x\tilde{u}|^2|(u^{r_2}-u_*)|^2+|\partial_xu^{r_2}|^2|(u_*-\tilde{u})|^2\}dx\\
\le& C\|\partial_x^2u^{r_2}\|^2+C\|\partial_xu^{r_2}\|_\infty^2\int_{\mathbb{R}_+}|\partial_x\tilde{u}|^2x^2dx+C\|\partial_xu^{r_2}\|_\infty^2\int_{\mathbb{R}_+}(u_*-\tilde{u})^2dx\\
\le& C\epsilon^{1+\frac{2}{q}}(1+t)^{-2(1-\frac{1}{q})}+C\tilde{\delta}(1+t)^{-2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}_+}|\hat{f}_x|^2dx \le& C\int_{\mathbb{R}_+}\left\{(|\partial_x^2\tilde{u}|^2+(\partial_x\tilde{u})^4)|(u^{r_2}-u_*)|^2+|\partial_x\tilde{u}|^2(\partial_xu^{r_2})^2+|\partial_x^2u^{r_2}|^2+(\partial_xu^{r_2})^4\right\}dx\\
\le& C\|\partial_x^2u^{r_2}\|^2+C\|\partial_xu^{r_2}\|_\infty^3\|\partial_xu^{r_2}\|_{L^1}+C\|\partial_xu^{r_2}\|_\infty^2\int_{\mathbb{R}_+}\left[(|\partial_x^2\tilde{u}|^2+(\partial_x\tilde{u})^4)x^2+|\partial_x\tilde{u}|^2\right]dx\\
\le& C\epsilon^{1+\frac{2}{q}}(1+t)^{-2(1-\frac{1}{q})}+C(\tilde{\delta}+\epsilon)(1+t)^{-2}.
\end{aligned}$$

$\square$

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